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MASS TRANSFER COOLING NEAR THE STAGNATION POINT

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SUMMARY

A simplified analysis is made of mass transfer cooling - that is, injection of a foreign gas - near the stagnation point for two-dimensional and axisymmetric bodies. The reduction in heat transfer is given in terms of the properties of the coolant gas and it is shown that the heat transfer may be reduced considerably by the introduction of a gas having appropriate thermal and diffusive properties. The mechanism by which heat transfer is reduced is discussed.

INTRODUCTION

The reduction of heat transfer near the stagnation point of a blunt body (two-dimensional or axisymmetric) is of primary importance when the body attains high velocity. One method of cooling is to introduce gas of high specific heat into the laminar boundary layer near the stagnation point and to allow this gas to flow over the nose of the body so that large amounts of heat are convected away from the nose.

A number of exact solutions based on various assumptions of fluid properties are available for stagnation-point heat transfer with no injection. (A comprehensive list of these references may be found in refs. 1, 2, and 3.) It has also been shown that the variation of the product of density and coefficient of viscosity across the boundary layer has an important effect on the heat transfer (ref. 4).

The boundary-layer equations for a binary mixture are well established (ref. 5) and several methods are available for the evaluation of the thermal and diffusive properties of a binary mixture (ref. 6).

Exact solutions (refs. 1 and 3) have recently been obtained for air-to-air injection near the stagnation point; these solutions have required lengthy numerical integration techniques and are necessarily limited to discrete values of the parameters involved (for example, the rate of injection of mass and the viscosity-law assumption).

The purpose of this paper is to present a simplified analysis by which the effect on the heat transfer of injection of air, or a foreign gas, may be found and, also, to show more clearly the important thermal and diffusive properties of the foreign gas if it is to be effective as a coolant. Application of the present method to ablation cooling is made in reference 7.

SYMBOLS

x   coordinate along wall

y   coordinate normal to wall

z   transformed y-coordinate

Y   arbitrary value of y outside boundary layer

Z   arbitrary value of z outside boundary layer

u   component of velocity in x-direction

v   component of velocity in y-direction

V   modified velocity component, $\dfrac{\rho v}{\rho_w}$

U   free-stream velocity in x-direction

C   constant in velocity distribution

T   temperature

$\bar{T}$ mean temperature (eq. (43))

W   concentration of foreign gas

$\bar{W}$ mean concentration (eq. (42))

ρ density of mixture

μ coefficient of viscosity

k   thermal conductivity of mixture

D_{12} coefficient of binary diffusivity

$\dot{m}$ rate of mass injection per unit area of wall

c_p specific heat at constant pressure

$\bar{c}_p$ mean specific heat (eq. (47))

N_{Nu} Nusselt number

R Reynolds number

N_{Pr} Prandtl number

N_{Sc} Schmidt number

q heat-transfer rate per unit area

δ_u velocity-boundary-layer thickness

δ_T thermal-boundary-layer thickness

δ_W concentration-boundary-layer thickness

K function of Prandtl number and Schmidt number (defined by eq. (40))

H_{eff} effective heat capacity of foreign gas

Subscripts:

e   external flow, near stagnation point

w   wall

c   coolant, far inside wall

0   no injection

1   foreign gas

2   air

b   body

DESCRIPTION OF THE BOUNDARY LAYER

The flow considered is that shown in figure 1. In the steady state the laminar boundary layer near the stagnation point is a thin layer of fluid in which the velocity, temperature, and concentration of the foreign gas vary rapidly from the external stream values to the wall values. In the neighborhood of the stagnation point the three super-imposed boundary layers (that is, the velocity, temperature, and concentration boundary layers) have constant, although different, thicknesses.

It is assumed that the component of velocity parallel to the wall is linear in x and that the normal component of velocity, the tempera-ture, the concentration of the foreign gas, and the properties of the mixture are all functions only of y, the distance normal to the wall. (This assumption results directly from the "similarity" nature of the flow and the absence of thermal and concentration gradients along the wall.) It is also assumed, consistent with the foregoing variation of velocity components, that the coolant gas is injected normally at the wall with a velocity independent of x.

When a given amount of coolant gas is injected, it diffuses through the boundary layer and is convected with the air, as a mixture, under the action of the pressure gradient imposed by the external flow and the shearing stress due to the presence of the wall. The concentration of foreign gas at the wall is uniquely determined by the rate of injec-tion. The extent of the boundary-layer shielding depends upon the spe-cific heat of the foreign gas and upon the wall temperature as well as the coolant and stagnation temperatures. The shielding also depends upon the manner in which the coolant gas diffuses through the boundary layer. The analysis shows the relative importance of these quantities.

ANALYSIS

The method to be used is as follows: the wall conditions for coolant injection are formulated in a simple manner (and the results are justified by a more detailed consideration of the flow within the porous wall presented in the appendix); use is then made of previous "exact" results for stagnation-point heat transfer with no injection in order to determine the gross characteristics of the thermal and viscous boundary layers near the stagnation point. These results are then used in a simple approximate integral method to determine the effect of injec-tion on the heat transfer to the wall.

The Coolant Flow

The boundary conditions at a porous wall through which a coolant is injected may be obtained very simply by ignoring the presence of the solid part of the wall; the justification for this is given in the appendix where more detailed considerations are made.

When the steady flow of coolant toward the surface of the wall is considered, the volume taken up by the solid wall being neglected, it is seen that there is a balance of diffusion and convection which governs the flow of mass and heat within the wall.

The transfer of mass is given simply by

$$\rho v = \rho_w v_w = \dot{m} \tag{1}$$

where $\dot{m}$ is constant.

The diffusion of air inward from the surface is balanced by the convection toward the surface:

$$-\rho D_{12} \frac{dW}{dy} \qquad = \qquad (1 - W)\dot{m} \tag{2}$$

Diffusion of air   Convection of air
from surface   toward surface

since W is the concentration of the foreign gas and $(1 - W)$ is that of air.

Similarly, the transfer of heat is given by

$$k \frac{dT}{dy} \qquad = \qquad c_{p,1}(T - T_c)\dot{m} \tag{3}$$

Diffusion of   Convection of heat
heat from   toward surface
surface

It is important to note here that, even though the specific heat of the mixture c_p is given by

$$c_p = c_{p,1}W + c_{p,2}(1 - W) \tag{4}$$

(with $c_{p,1}$ and $c_{p,2}$ constant), the value used correctly in equation (3) is $c_{p,1}$; in unit time the heat transferred at any point y in a

direction away from the surface is sufficient to raise an amount of foreign gas of specific heat $c_{p,1}$ through the temperature range $T - T_c$.

Evaluating equations (2) and (3) at the wall surface gives the following boundary conditions which are required for the solution of the boundary-layer equations:

$$-\left[\rho D_{12} \frac{dW}{dy}\right]_W = (1 - W_W)\dot{m} \tag{5}$$

$$\left[k \frac{dT}{dy}\right]_W = c_{p,1}(T_W - T_c)\dot{m} \tag{6}$$

Detailed study of the flow within the porous wall yields the same boundary conditions. (See appendix.)

The Boundary-Layer Integral Equations

The integral equations which describe the transfer of mass and heat in the boundary layer are derived in a simple way without reference to the general differential equations.

Consider a small rectangle of height Y (Y greater than any boundary-layer thickness) and length Δx, in which ρ, W, and T are independent of x, near the stagnation point of a two-dimensional body. (See fig. 2.) The continuity of mass may be expressed as follows:

$$-\rho(Y)v(Y) \Delta x \quad + \quad \dot{m} \Delta x \quad = \quad \left[\int_0^Y \rho u \, dy\right]_{x+\Delta x} \quad - \quad \left[\int_0^Y \rho u \, dy\right]_x$$

Flow in from   Injection   Flow out at   Flow in
external stream $x + \Delta x$ at x

This equation, in the limit as $\Delta x \rightarrow 0$, becomes

$$-\rho(Y)v(Y) + \dot{m} = \frac{d}{dx} \int_0^Y \rho u \, dy \tag{7}$$

When the form $U = Cx$ of the external velocity is used and it is noted that

$$\frac{d}{dx} \int_0^Y \rho u \, dy = \int_0^Y \rho \frac{du}{dx} \, dy$$

and

$$u = C \frac{u}{U} x$$

where $\frac{u}{U}$ is independent of x, then equation (7) reduces to

$$-\rho(Y)v(Y) + \dot{m} = C \int_0^Y \rho \frac{u}{U} \, dy \tag{8}$$

Since the density ρ is a function of concentration and temperature, it is convenient to introduce the quantities

$$\left. \begin{aligned} V &= \frac{\rho v}{\rho_w} \\[2mm] dz &= \frac{\rho}{\rho_w} \, dy \end{aligned} \right\} \tag{9}$$

so that equation (8) becomes finally

$$-\rho_w V(Z) + \dot{m} = C\rho_w \int_0^Z \frac{u}{U} \, dz \tag{10}$$

Similarly, the equation for the transfer of foreign gas becomes

$$\dot{m} \qquad = \qquad C\rho_w \int_0^Z W \frac{u}{U} \, dz \tag{11}$$

Gas injected   Gas convected

The equation of heat transfer is written

$$-c_{p,2}(T_e - T_w)\rho_w V(Z) \quad - \quad \left(k\,\frac{dT}{dy}\right)_w \quad = \quad C\rho_w \int_0^Z c_p(T - T_w)\,\frac{u}{U}\,dz$$

(12)

| Convection of heat | Transfer of | Convection of heat |
| from external stream | heat to wall | in boundary layer |

The quantity $V(Z)$ may be eliminated from equation (12) by use of equations (4), (10), and (11); thus,

$$C\rho_w \int_0^Z c_p(T_e - T)\,\frac{u}{U}\,dz = \left(k\,\frac{dT}{dy}\right)_w + c_{p,1}(T_e - T_w)\dot{m}$$

(13)

Alternatively, by use of the boundary condition given by equation (6), equation (13) has the form

$$C\rho_w \int_0^Z c_p(T_e - T)\,\frac{u}{U}\,dz = c_{p,1}(T_e - T_c)\dot{m}$$

(14)

The equations for transfer of heat and mass near the axisymmetric stagnation point can be derived in a similar manner and are identical with equations (10) to (14) where x is again the distance from the stagnation point along the surface but $z = 2\displaystyle\int_0^y \frac{\rho}{\rho_w}\,dy$.

Heat-Transfer Relations for No Injection

The rate of heat transfer is first given in terms of Nusselt number $N_{Nu,w}$, Reynolds number R_w, and Prandtl number $N_{Pr,w}$ for which the usual definitions near the stagnation point are used:

$$N_{Nu,w} = \frac{x}{T_e - T_w}\left(\frac{dT}{dy}\right)_w$$

$$R_w = \frac{\rho_w U x}{\mu_w} = \frac{C\rho_w}{\mu_w}\,x^2$$

and

$$N_{Pr,w} = \frac{\mu_w}{R_w} c_{p,2}$$

The rate of heat transfer to the wall with no injection q_0 is now

$$q_0 = \left(k \frac{dT}{dy}\right)_w = c_{p,2}(T_e - T_w)\left(\rho_w \mu_w C\right)^{1/2} \frac{1}{N_{Pr,w}}\left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0 \qquad (15)$$

Exact evaluation of the dimensionless parameter $\left(\dfrac{N_{Nu,w}}{R_w^{1/2}}\right)_0$ has been

made by many investigators (for example, refs. 1, 2, and 4). The following formulas were found to be in good agreement with previous exact results (ref. 1) under the assumption of constant $\rho\mu$:

For the axisymmetric body,

$$\left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0 = \left[0.765 - 0.065\left(1 - \frac{T_w}{T_e}\right)\right]N_{Pr,w}^{0.4} \qquad (16)$$

and for the two-dimensional body,

$$\left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0 = \left[0.570 - 0.065\left(1 - \frac{T_w}{T_e}\right)\right]N_{Pr,w}^{0.4} \qquad (17)$$

Alternative expressions, based on the solutions presented in reference 4 in which $\rho\mu$ is not constant, are as follows:

For the axisymmetric body,

$$\left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0 = 0.765\left(\frac{\rho_e \mu_e}{\rho_w \mu_w}\right)^{0.4} N_{Pr,w}^{0.4} \qquad (18)$$

and for the two-dimensional body,

$$\left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0 = 0.570\left(\frac{\rho_e\mu_e}{\rho_w\mu_w}\right)^{0.4} N_{Pr,w}^{0.4} \tag{19}$$

Boundary-Layer Thickness Without Injection

Use is now made of these results (eqs. (16) to (19)) to determine the thickness $\delta_{u,0}$ of the velocity boundary layer without injection. Equation (13) with $\dot{m} = 0$ is written in the form

$$\frac{1}{\delta_{u,0}}\int_0^Z \frac{T_e - T}{T_e - T_w}\frac{u}{U}\,dz = \frac{1}{\delta_{u,0}}\left(\frac{\mu_w}{\rho_w C}\right)^{1/2}\frac{1}{N_{Pr,w}}\left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0 \tag{20}$$

and linear profiles are assumed for velocity and temperature in order to evaluate the integral in equation (20); that is,

$$\left.\begin{aligned}
\frac{u}{U} &= \frac{z}{\delta_{u,0}}\\[2ex]
\frac{T_e - T}{T_e - T_w} &= 1 - \frac{z}{\delta_{T,0}}
\end{aligned}\right\} \tag{21}$$

where $\delta_{T,0}$ is the thickness of the thermal boundary layer. The left-hand side of equation (20) is then approximately $\frac{1}{6}\left(\frac{\delta_{T,0}}{\delta_{u,0}}\right)^2$ and equation (20) reduces to

$$\frac{1}{6}\left(\frac{\delta_{T,0}}{\delta_{u,0}}\right)^2 = \frac{1}{\delta_{u,0}}\left(\frac{\mu_w}{\rho_w C}\right)^{1/2}\frac{1}{N_{Pr,w}}\left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0 \tag{22}$$

If the value of $\left(\dfrac{N_{Nu,w}}{R_w^{1/2}}\right)_0$ when $N_{Pr,w} = 1$ is signified by the super-

script (1), then equations (16) to (19) show that

$$\left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0 = \left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0^{(1)} N_{Pr,w}^{0.4} \tag{23}$$

This equation shows the effect of the Prandtl number on the heat-transfer coefficient. When $N_{Pr,w} = 1$, the thermal and viscous diffusive effects are similar so that $\dfrac{\delta_{T,0}}{\delta_{u,0}} = 1$; thus, from equation (22)

$$\delta_{u,0}\left(\frac{\rho_w C}{\mu_w}\right)^{1/2} = 6\left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0^{(1)} \tag{24}$$

The nondimensional thickness $\delta_{u,0}\left(\dfrac{\rho_w C}{\mu_w}\right)^{1/2}$ is now assumed to be inde-

pendent of the Prandtl number (since the momentum equation is only weakly dependent on the Prandtl number); thus, when equations (23) and (24) are inserted into equation (22), the following simple result is obtained:

$$\frac{\delta_{T,0}}{\delta_{u,0}} = N_{Pr,w}^{-0.3} \tag{25}$$

This result is analogous to that for a flat plate (ref. 8):

$$\frac{\delta_T}{\delta_u} = N_{Pr,w}^{-1/3}$$

Injection of Air

Consider now the effect of injection of air into the boundary layer. The boundary layer becomes thicker because of the increase in mass flow which results in reduced gradients in the boundary layer. In particular, the heat transfer to the wall is reduced since the heat convected parallel to the wall in the boundary layer is increased.

The effect of air injection on the boundary-layer thickness is given by considering the mass flow in unit time per unit area, as follows:

$$C\rho_w \int_0^{\delta_u} \frac{u}{U}\, dz \quad = \quad C\rho_w \int_0^{\delta_{u,0}} \frac{u}{U}\, dz \quad + \quad \dot{m}$$

| Mass flow with | Mass flow with | Mass |
| injection | no injection | injected |

which gives

$$\frac{1}{2}\, \delta_u = \frac{1}{2}\, \delta_{u,0} + \frac{\dot{m}}{C\rho_w} \tag{26}$$

when linear profiles are assumed for $\frac{u}{U}$ $\left(\text{that is, } \frac{u}{U} = \frac{z}{\delta_{u,0}} \text{ (no injec-}\right.$ tion) and $\frac{u}{U} = \frac{z}{\delta_u}$ (injection)$\Big)$.

It is assumed that the relation given by equation (25) is still true since the ratio of thermal-boundary-layer thickness to velocity-boundary-layer thickness depends on diffusive processes, that is, on the Prandtl number.

With $c_{p,1} = c_{p,2}$ equation (13) may be written by use of equation (15) in the form

$$\frac{1}{\delta_u} \int_0^Z \frac{T_e - T}{T_e - T_w} \frac{u}{U}\, dz = \frac{1}{\delta_u} \left(\frac{\mu_w}{\rho_w C}\right)^{1/2} \left[\frac{1}{N_{Pr,w}} \frac{N_{Nu,w}}{R_w^{1/2}} + \frac{\dot{m}}{(\rho_w \mu_w C)^{1/2}}\right]$$

Linear profiles for temperature and velocity (eqs. (21)) and the expressions for δ_u (eq. (26)) and $\delta_{u,0}$ (eq. (24)) are used to simplify this equation, with the following result:

$$\frac{1}{N_{Pr,w}} \frac{N_{Nu,w}}{R_w^{1/2}} = \frac{1}{N_{Pr,w}} \left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0 - \left(1 - \frac{1}{3} N_{Pr,w}^{-0.6}\right) \frac{\dot{m}}{(\rho_w \mu_w C)^{1/2}} \quad (27)$$

In terms of the rate of heat transfer q, equation (27) has the form

$$q = q_0 - \left(1 - \frac{1}{3} N_{Pr,w}^{-0.6}\right) c_{p,2} (T_e - T_w) \dot{m} \quad (28)$$

$\quad\quad\quad\quad$ Heat transfer $\quad\quad\quad\quad\quad\quad\quad\quad$ Boundary-layer shielding
$\quad\quad\quad\quad$ for no $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ by convection
$\quad\quad\quad\quad$ injection

The quantity $\left(1 - \frac{1}{3} N_{Pr,w}^{-0.6}\right)(T_e - T_w)$ may be interpreted as the average temperature rise in the boundary layer of the mass introduced at the wall surface. An alternative expression of this heat balance, obtained by using the boundary condition given by equation (6), is

$$q_0 = c_{p,2} \left[(T_w - T_c) + \left(1 - \frac{1}{3} N_{Pr,w}^{-0.6}\right)(T_e - T_w)\right] \dot{m} \quad (29)$$

When q_0 is expressed in terms of dimensionless quantities by use of equations (15) and (23), equation (29) becomes

$$\frac{T_e - T_w}{T_e - T_c} = \frac{N_{Pr,w}^{0.6} \dfrac{\dot{m}}{(\rho_w \mu_w C)^{1/2}}}{\left(\dfrac{N_{Nu,w}}{R_w^{1/2}}\right)_0^{(1)} + \dfrac{1}{3} \dfrac{\dot{m}}{(\rho_w \mu_w C)^{1/2}}} \quad (30)$$

Equation (30) may be used to determine the rate of injection required to maintain the wall temperature T_w at a given desired value when T_e and T_c are specified.

Before proceeding to considerations of injection of foreign gas, the results given by equation (27) are compared with available exact solutions as a check on the validity of the several simplifying assumptions that have been made.

Firstly, under the assumption of constant $\rho\mu$, equations (16) (eq. (17) for the two-dimensional flow) and (27) are used to give $\dfrac{N_{Nu,w}}{R_w^{1/2}}$ as a function of the dimensionless rate of mass injection $\dfrac{\dot{m}}{(\rho_w\mu_w C)^{1/2}}$. A comparison of these results with the results of references 1 and 3 shows very good agreement except for the extreme rates of injection with Prandtl number equal to unity. (See figs. 3, 4, and 5.)

Secondly, for variable $\rho\mu$, equations (18) (or eq. (19)) and (27) are used and the results are compared with those of references 3 and 4. For a Prandtl number of 0.71 and no injection reference 4 gives, for the axisymmetric case,

$$\left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0 = 0.67\left(\frac{\rho_e\mu_e}{\rho_w\mu_w}\right)^{0.4}$$

and the present report gives (from eq. (18))

$$\frac{N_{Nu,w}}{R_w^{1/2}} = 0.667\left(\frac{\rho_e\mu_e}{\rho_w\mu_w}\right)^{0.4}$$

A comparison of the results obtained by use of equation (18) with the results of reference 3 in which the Sutherland viscosity law was used is presented in figure 6.

The effects of injection are also compared when $\rho\mu$ is variable; again good agreement is found except for the high injection rates. (See fig. 5.)

With this justification of the assumptions that have been made, the method is now extended to take account of the additional effects which result from the injection of a foreign gas (that is, gas having properties different from those of air).

Injection of Foreign Gas

The mechanism by which air diffuses through the boundary layer in a binary mixture due to the concentration gradient is exactly analogous to that by which heat diffuses in the absence of viscous dissipation for air-to-air injection. In this analogy the coefficient of binary diffusivity D_{12} corresponds to the thermal diffusivity $\dfrac{k}{\rho c_{p,2}}$ and the Schmidt number $N_{Sc} = \dfrac{\mu}{\rho D_{12}}$ corresponds to the Prandtl number $N_{Pr} = \dfrac{\mu c_{p,2}}{k}$.

The analogy is easily seen when the gas-transfer equation (11) is rewritten as

$$C\rho_w \int_0^Z \frac{W}{W_w} \frac{u}{U} \, dz = \frac{\dot{m}}{W_w} \tag{31}$$

and compared with the heat-transfer equation for air-to-air injection,

$$C\rho_w \int_0^Z \frac{T_e - T}{T_e - T_w} \frac{u}{U} \, dz = \frac{\dot{m}}{\dfrac{T_e - T_w}{T_e - T_c}} \tag{32}$$

The boundary condition at the wall is

$$-\left[\rho D_{12} \frac{dW}{dy}\right]_w = (1 - W_w)\dot{m} \tag{33}$$

which is compared with

$$\left[\rho \frac{k}{\rho c_{p,2}} \frac{dT}{dy}\right]_w = (T_w - T_c)\dot{m} \tag{34}$$

The analogous quantities are presented in the following table:

Binary diffusion	Thermal diffusion
W	$\dfrac{T_e - T}{T_e - T_c}$
$1 - W$	$\dfrac{T - T_c}{T_e - T_c}$
W_w	$\dfrac{T_e - T_w}{T_e - T_c}$
D_{12}	$\dfrac{k}{\rho c_{p,2}}$
N_{Sc}	N_{Pr}

It may be verified that all the boundary conditions at the wall, in the external flow, and at large distances into the interior of the wall agree with the foregoing analogy.

Because of the foregoing comparison the following results may be written immediately: from equation (25),

$$\frac{\delta_W}{\delta_u} = N_{Sc,w}^{-0.3} \tag{35}$$

where δ_W is the thickness of the concentration boundary layer, and from equation (30)

$$W_w = \frac{N_{Sc,w}^{0.6}\dfrac{\dot{m}}{(\rho_w\mu_w C)^{1/2}}}{\left(\dfrac{N_{Nu,w}}{R_w^{1/2}}\right)_0^{(1)} + \dfrac{1}{3}\dfrac{\dot{m}}{(\rho_w\mu_w C)^{1/2}}} \tag{36}$$

The value of $\left(\dfrac{N_{Nu,w}}{R_w^{1/2}}\right)_0^{(1)}$ to be used in equation (36) is necessarily that given by equation (18) or equation (19) since $\rho\mu$ is now a variable quantity and depends on the concentration of foreign gas in the boundary layer.

When the gas injected is different from air, the amount of shielding by convection in the boundary layer depends critically upon the specific heat of the gas $c_{p,1}$.

The heat-transfer equation (13) is rewritten, by use of equations (4) and (6), as

$$C\rho_w \int_0^z \left[c_{p,2} + (c_{p,1} - c_{p,2})W\right](T_e - T)\frac{u}{U}\,dz = c_{p,1}\dot{m}(T_e - T_c) \quad (37)$$

and the linear profile

$$\frac{W}{W_w} = 1 - \frac{z}{\delta_W} \quad (38)$$

is assumed.

Substitution of equations (21), equation (38), and the definition of N_{Nu} into equation (37) gives results similar in form to those for air-to-air injection (eq. (27)):

$$\frac{1}{N_{Pr,w}}\frac{N_{Nu,w}}{R_w^{1/2}} = \frac{1}{N_{Pr,w}}\left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0 - \left(1 - \frac{1}{3}N_{Pr,w}^{-0.6}\right)\frac{\dot{m}}{(\rho_w\mu_wC)^{1/2}} -$$

$$\left(\frac{c_{p,1}}{c_{p,2}} - 1\right)\left(1 - KN_{Sc,w}^{0.6}\right)\frac{\dot{m}}{(\rho_w\mu_wC)^{1/2}} \quad (39)$$

where

$$K = 6 \frac{1}{\delta_u} \int_0^Z \frac{W}{W_w} \frac{T_e - T}{T_e - T_w} \frac{u}{U} \, dz \tag{40}$$

and is a function of $N_{Pr,w}$ and $N_{Sc,w}$.

Equation (39) may be expressed in the alternative form

$$q = q_0 - \left[c_{p,1} \frac{1 - KN_{Sc,w}^{0.6}}{1 - \frac{1}{3} N_{Pr,w}^{-0.6}} + c_{p,2} \frac{KN_{Sc,w}^{0.6} - \frac{1}{3} N_{Pr,w}^{-0.6}}{1 - \frac{1}{3} N_{Pr,w}^{-0.6}} \right] \left(1 - \frac{1}{3} N_{Pr,w}^{-0.6} \right) (T_e - T_w) \dot{m} \tag{41}$$

The second term on the right-hand side of equation (41) has the form

$$\left[c_{p,1} \overline{W} + c_{p,2} (1 - \overline{W}) \right] (\overline{T} - T_w) \dot{m}$$

where $\overline{W}$, the mean concentration of the foreign gas in the boundary layer, is

$$\overline{W} = \frac{1 - KN_{Sc,w}^{0.6}}{1 - \frac{1}{3} N_{Pr,w}^{-0.6}} \tag{42}$$

and where $\overline{T}$, the mean temperature in the boundary layer, is

$$\overline{T} = T_w + \left(1 - \frac{1}{3} N_{Pr,w}^{-0.6} \right) (T_e - T_w) \tag{43}$$

In terms of these mean values equation (41) becomes

$$q \quad = \quad q_0 \quad - \quad \left[c_{p,1} \overline{W} + c_{p,2} (1 - \overline{W}) \right] (\overline{T} - T_w) \dot{m} \tag{44}$$

Heat   Heat transfer   Boundary-layer shielding
transfer   for no
                injection

Thus, in order to achieve the maximum amount of shielding for given temperature conditions and mass injection, $c_{p,1}$ should be made as large as possible and, in addition, $\overline{T}$ should be made as large as possible; if $c_{p,1} > c_{p,2}$ then $\overline{W}$ should be large, and if $c_{p,1} < c_{p,2}$ then $\overline{W}$ should be small. The variations with $N_{Pr,w}$ and $N_{Sc,w}$ of $\dfrac{\overline{T} - T_w}{T_e - T_w}$ and $\overline{W}$ are shown in figures 7 and 8, respectively. It is seen that $\overline{W}$ increases as $N_{Sc,w}$ decreases and that $\dfrac{\overline{T} - T_w}{T_e - T_w}$ increases as $N_{Pr,w}$ increases.

Maximum shielding in the boundary layer is therefore achieved when, for $c_{p,1} > c_{p,2}$, $N_{Pr,w}$ is large and $N_{Sc,w}$ is small and, for $c_{p,1} < c_{p,2}$, $N_{Pr,w}$ is large and $N_{Sc,w}$ is large.

These diffusive effects on the shielding are explained qualitatively in the following way (fig. 9): the convective shielding is most efficient when the gas of higher specific heat is transported in the regions of highest temperature and velocity, that is, in the part of the boundary layer farthest from the wall.

Equation (25) shows that when $N_{Pr,w}$ is large the velocity boundary layer is thicker than the thermal boundary layer and thus more of the hot gas mixture is convected. When $c_{p,1} > c_{p,2}$ and $N_{Sc,w}$ is small, equation (35) shows that the concentration-boundary-layer thickness is greater than the velocity-boundary-layer thickness - that is, the foreign gas of higher specific heat diffuses quickly through the boundary layer before being convected. On the other hand, when $c_{p,1} < c_{p,2}$ and $N_{Sc,w}$ is large, the foreign gas remains near the wall and displaces the air of higher specific heat into the hot fast region where it is convected.

An important consideration in any cooling system is the weight penalty which is incurred in order to maintain the wall at a desirable temperature. The relation between the rate of mass flow $\dot{m}$ and the temperatures T_e, T_w, and T_c is obtained by substitution of the expressions

$$q = c_{p,1}(T_w - T_c)\dot{m}$$

and

$$q_0 = c_{p,2}(T_e - T_w)(\rho_w \mu_w C)^{1/2} \frac{1}{N_{Pr,w}} \left(\frac{N_{Nu,w}}{R_w^{1/2}}\right)_0$$

(from eqs. (6) and (13), respectively) into equation (41). This substitution results in the following expressions $\left(\text{when either equation (18)}\right.$ or equation (19) is used for $\left.\left(\dfrac{N_{Nu,w}}{R_w^{1/2}}\right)_0\right)$:

For axisymmetric flow,

$$\dot{m} = \frac{0.765\left(\dfrac{\rho_e \mu_e}{\rho_w \mu_w}\right)^{0.4} N_{Pr,w}^{-0.6}(\rho_w \mu_w C)^{1/2} c_{p,2}(T_e - T_w)}{c_{p,1}(T_w - T_c) + \bar{c}_p(\bar{T} - T_w)} \tag{45}$$

and for two-dimensional flow,

$$\dot{m} = \frac{0.570\left(\dfrac{\rho_e \mu_e}{\rho_w \mu_w}\right)^{0.4} N_{Pr,w}^{-0.6}(\rho_w \mu_w C)^{1/2} c_{p,2}(T_e - T_w)}{c_{p,1}(T_w - T_c) + \bar{c}_p(\bar{T} - T_w)} \tag{46}$$

where

$$\bar{c}_p = c_{p,1}\bar{W} + c_{p,2}(1 - \bar{W}) \tag{47}$$

Probably the most important parameter is the total effective heat capacity H_{eff} of the coolant; this parameter is a measure of the total heat absorbed when unit mass of the coolant is used and is defined as

$$H_{eff} = \frac{q_0}{\dot{m}} = c_{p,1}(T_w - T_c) + \bar{c}_p(\bar{T} - T_w) \tag{48}$$

METHOD OF CALCULATION OF RATE OF MASS INJECTION

The problem of most interest from practical considerations is that of determining the rate of mass injection $\dot{m}$ necessary to achieve a desired wall temperature T_w when the stagnation temperature T_e and the coolant temperature T_c are given. This calculation is made complicated by the dependence of the mean specific heat $\bar{c}_p$ (required for eq. (45) or eq. (46)) on the rate of mass injection; both these quantities depend on the concentration W_w. (See eqs. (47), (42), and (35).)

The following method, however, gives results with relatively little computation:

(1) The parameters $N_{Pr,w}$ and $N_{Sc,w}$ are found in terms of W_w for the particular binary mixture under consideration. (Simple empirical methods are given in ref. 6.)

(2) Equation (43)

$$\frac{\bar{T} - T_w}{T_e - T_w} = 1 - \frac{1}{3} N_{Pr,w}^{-0.6}$$

is used to give $\bar{T}$ (fig. 7).

(3) Equation (42)

$$\bar{W} = \frac{1 - K N_{Sc,w}^{0.6}}{1 - \frac{1}{3} N_{Pr,w}^{-0.6}}$$

is used to give $\bar{W}$ (fig. 8).

(4) Equation (47)

$$\bar{c}_p = c_{p,1}\bar{W} + c_{p,2}(1 - \bar{W})$$

then gives $\bar{c}_p$ (fig. 10).

(5) A plot of $\dfrac{\dot{m}}{(\rho_w\mu_w C)^{1/2}}\left(\dfrac{\rho_w\mu_w}{\rho_e\mu_e}\right)^{0.4}$ against W_w can now be made from the results of the substitution of equations (43), (42), and (47) into either equation (45) or equation (46).

(6) A second plot of $\dfrac{\dot{m}}{(\rho_w\mu_w C)^{1/2}}\left(\dfrac{\rho_w\mu_w}{\rho_e\mu_e}\right)^{0.4}$ against W_w can be obtained as a result of converting equation (36) into the following form for the axisymmetric case:

$$\frac{\dot{m}}{(\rho_w\mu_w C)^{1/2}}\left(\frac{\rho_w\mu_w}{\rho_e\mu_e}\right)^{0.4} = \frac{0.765 W_w}{N_{Sc,w}{}^{0.6} - \frac{1}{3}W_w} \tag{49}$$

or into the following form for the two-dimensional case:

$$\frac{\dot{m}}{(\rho_w\mu_w C)^{1/2}}\left(\frac{\rho_w\mu_w}{\rho_e\mu_e}\right)^{0.4} = \frac{0.570 W_w}{N_{Sc,w}{}^{0.6} - \frac{1}{3}W_w} \tag{50}$$

The intersection of the curves found in steps (5) and (6) gives W_w and

$$\frac{\dot{m}}{(\rho_w\mu_w C)^{1/2}}\left(\frac{\rho_w\mu_w}{\rho_e\mu_e}\right)^{0.4}.$$

(7) Now, $\rho_w\mu_w$ can be found from W_w and T_w, and $\rho_e\mu_e$ from external conditions; thus, $\dot{m}$ can be calculated.

Step (7) may be omitted if the approximation

$$\left(\frac{\rho_w\mu_w}{\rho_e\mu_e}\right)^{0.4} \approx \left(\frac{\rho_w\mu_w}{\rho_e\mu_e}\right)^{1/2}$$

is made, in which case $\dfrac{\dot{m}}{(\rho_e\mu_e C)^{1/2}}$ is given directly by steps (5) and (6).

In order to see the main effect of $\dfrac{c_{p,1}}{c_{p,2}}$ on the mass-flow require-

ments for given temperature parameter $\dfrac{T_w - T_c}{T_e - T_w}$, values of the Prandtl

number and Schmidt number equal to unity are taken.

Thus, from equation (43)

$$\frac{\overline{T} - T_w}{T_e - T_w} = \frac{2}{3}$$

from equation (40) $K = \frac{1}{2}$, and from equation (42) $\overline{W} = \frac{3}{4}$. Using this

value for $\overline{W}$ in equation (47) gives

$$\overline{c}_p = \frac{3}{4}\, c_{p,1} + \frac{1}{4}\, c_{p,2}$$

Then, equation (45) for axisymmetric flow becomes

$$\frac{\dot{m}}{\left(\rho_e \mu_e C\right)^{1/2}} \approx \frac{\dot{m}}{\left(\rho_w \mu_w C\right)^{1/2}}\left(\frac{\rho_w \mu_w C}{\rho_e \mu_e}\right)^{0.4} = \frac{0.765}{\dfrac{c_{p,1}}{c_{p,2}}\left(\dfrac{T_w - T_c}{T_e - T_w} + \dfrac{1}{2}\right) + \dfrac{1}{6}}$$

and equation (46) for two-dimensional flow becomes

$$\frac{\dot{m}}{\left(\rho_e \mu_e C\right)^{1/2}} \approx \frac{0.570}{\dfrac{c_{p,1}}{c_{p,2}}\left(\dfrac{T_w - T_c}{T_e - T_w} + \dfrac{1}{2}\right) + \dfrac{1}{6}}$$

The results are shown in figure 11. It is seen that the required

mass flow is reduced by a large factor for the higher values of $\dfrac{c_{p,1}}{c_{p,2}}$.

This is especially true for the higher heating rates $\left(\text{as } \dfrac{T_w - T_c}{T_e - T_w} \to 0\right)$

inasmuch as the shielding by heat convection in the boundary layer is
large.

CONCLUDING REMARKS

An approximate analysis has been presented whereby the reduction in heat transfer near the stagnation point may be calculated with little difficulty when the coolant properties are known. The agreement with available exact solutions for air-to-air injection is extremely good in view of the simple approach employed, and the qualitative trends of the results for foreign-gas injection in explaining the shielding mechanism suggest that the approximate analysis will generally give reliable results. It is expected that the simplified calculation which shows the dependence of the coolant mass requirement on wall temperature and coolant specific heat will provide a good estimate for engineering purposes.

The conclusions of the analysis are summarized briefly as follows:

Maximum boundary-layer shielding is achieved when the gas of higher specific heat is convected in the hot fast-moving part of the boundary layer farthest from the wall; this requires that the coolant gas introduced have

(a) High specific heat compared with that of air

(b) High binary diffusivity (that is, small Schmidt number)

(c) Large Prandtl number.

Of these three properties the first is the most important, whereas the diffusive effects are of secondary importance.

Langley Aeronautical Laboratory,
    National Advisory Committee for Aeronautics,
        Langley Field, Va., July 3, 1958.

APPENDIX

FLOW WITHIN THE POROUS WALL

For the purpose of the present analysis it is assumed that the wall is uniformly porous and presents, to the stream, an area A of solid material and an area B across which coolant gas is transported.

The manner in which gas is transported through the porous material is given by

$$-\rho D_{12} \frac{dW}{dy} B \quad + \quad \rho v W B \quad = \quad \dot{m}(A + B) \tag{A1}$$

Diffusion   Convection   Total gas
toward surface   toward   transported
                   surface

since $\dot{m}$ is the rate of mass introduced per unit area of body surface.

The air contained in the mixture is transported according to the following equation:

$$-\rho D_{12} \frac{dW}{dy} B \quad = \quad \rho v (1 - W) B \tag{A2}$$

Diffusion of air   Convection
from surface   toward surface

since there is no net transport of air into the body in the steady state. From equations (A1) and (A2),

$$\rho v B = \dot{m}(A + B) \tag{A3}$$

The equation for transfer of heat within the porous wall is

$$(k_b A + kB)\frac{dT}{dy} \quad = \quad c_{p,l}(T - T_c)\dot{m}(A + B) \tag{A4}$$

Conduction into   Rate heat absorbed
solid and coolant   by coolant

Equation (A4) may be written

$$\bar{k} \frac{dT}{dy} = c_{p,1}(T - T_c)\dot{m} \tag{A5}$$

where $\bar{k}$ is a mean value of the effective thermal conductivity $\frac{k_b A + kB}{A + B}$. Similarly, equation (A2) may be written (by using eq. (A3)) as

$$\overline{\rho D}_{12} \frac{dW}{dy} = \dot{m}W \tag{A6}$$

where $\overline{\rho D}_{12}$ is a mean value of $\frac{\rho D_{12}B}{A + B}$.

Equations (A5) and (A6) have solutions (similar in nature) of the form

$$T = T_c + (T_w - T_c)e^{\frac{\dot{m}c_{p,1}y}{\bar{k}}} \tag{A7}$$

which satisfies the condition $T \to T_c$ as $y \to -\infty$, and

$$W = 1 - (1 - W_w)e^{\frac{\dot{m}}{\overline{\rho D}_{12}}y} \tag{A8}$$

A characteristic thickness θ_T may be defined as

$$\theta_T = \int_0^{-\infty} \frac{T - T_c}{T_w - T_c} \, dy = \frac{\bar{k}}{\dot{m}c_{p,1}} \tag{A9}$$

in which the temperature changes from its value at the surface T_w to its value at the far interior T_c.

Similarly, the concentration W changes from W_w at the surface to the value 1 in a layer of thickness θ_W (within the porous wall) where

$$\theta_W = \int_0^{-\infty} \frac{1 - W}{1 - W_w} \, dy = \frac{\overline{\rho D}_{12}}{\dot{m}} \tag{A10}$$

It is seen that both thicknesses depend on the ratio $\frac{B}{A}$. It is desirable to limit the relatively high temperature region within the wall to a narrow region near the surface; the specific heat c_p must be large and $\bar{k}$ must be small, that is, both k_b and k should be small. Considerations of the boundary-layer flow outside the wall also lead to the conclusion that the specific heat $c_{p,l}$ should be large and k should be small (so that the Prandtl number may be as large as possible). Thus, these requirements result in two desirable features:

(a) Confinement of the high temperatures within the body to a narrow region near the surface and

(b) Maximum convection of heat away from the stagnation point (as shown in the body of the report)

The conditions at the surface as it is approached from within the wall are given as follows:

From equations (A2) and (A3)

$$-\left(\rho D_{12} \frac{dW}{dy}\right)_{w-} B = (1 - W_{w-})\dot{m}(A + B) \tag{A11}$$

and from equation (A4)

$$(k_b A + kB)\left(\frac{dT}{dy}\right)_{w-} = c_{p,l}(T_w - T_c)\dot{m}(A + B) \tag{A12}$$

where the subscript $w-$ denotes that the surface is approached through negative values of y. The rate of heat transfer from the stream boundary layer is $\left(k \dfrac{dT}{dy}\right)_w (A + B)$ so that the boundary condition (eq. (A12)) becomes

$$\left(k \frac{dT}{dy}\right)_w = c_{p,l}(T_w - T_c)\dot{m} \tag{A13}$$

Thus, there is a discontinuity in $\frac{dT}{dy}$ at the surface when $k_w \neq k_b$.

Similarly, the condition of no net transfer of air to the body is given by

$$-\left(\rho D_{12} \frac{dW}{dy}\right)_W (A + B) = (1 - W_W)\rho_W v_W (A + B) \qquad (A14)$$

obtained when the surface is approached through positive values of y. Since

$$\rho_{W-} v_{W-} = \dot{m} = \rho_W v_W$$

equation (A14) may be written as

$$-\left(\rho D_{12} \frac{dW}{dy}\right)_W = (1 - W_W)\dot{m} \qquad (A15)$$
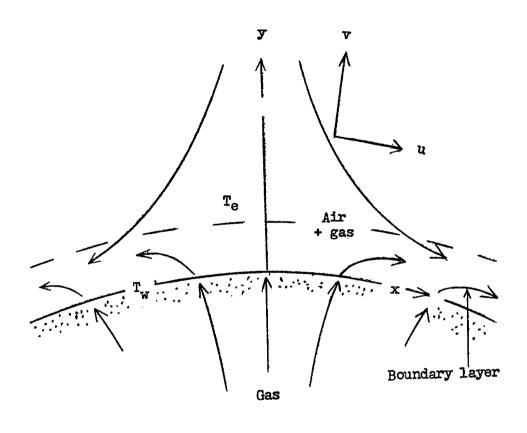
It is noted that v is discontinuous since there is a discontinuity in the area available for convection given by the ratio $\frac{B}{A + B}$. The concentration gradient $\frac{dW}{dy}$ is also discontinuous for the same reason; this is expected by analogy with the temperature gradient $\frac{dT}{dy}$. Equations (A13) and (A15) are seen to be identical with equations (3) and (2) when evaluated at the wall.

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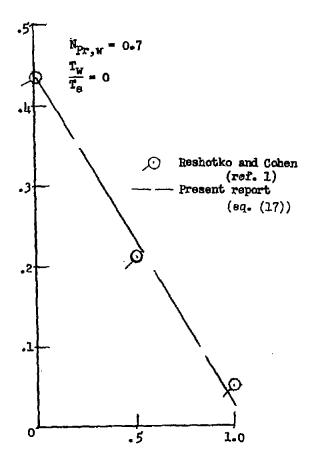Figure 1.- Flow configuration.
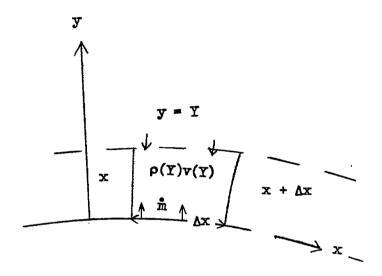
Figure 2.- Boundary-layer mass balance.
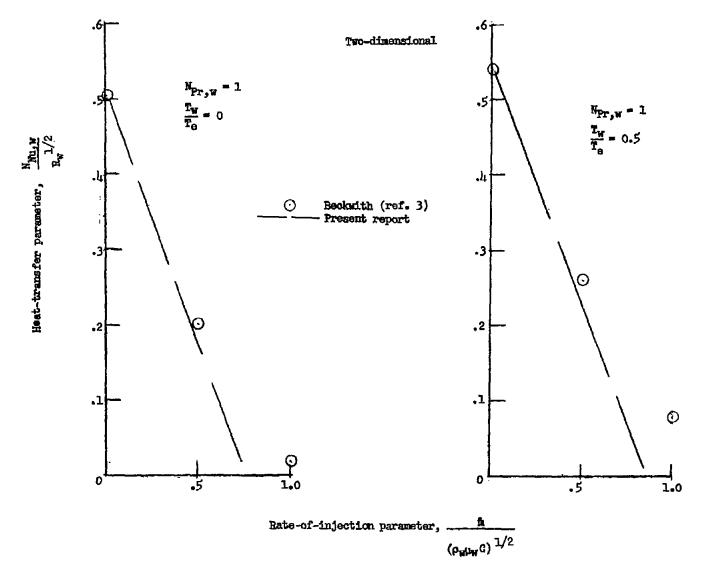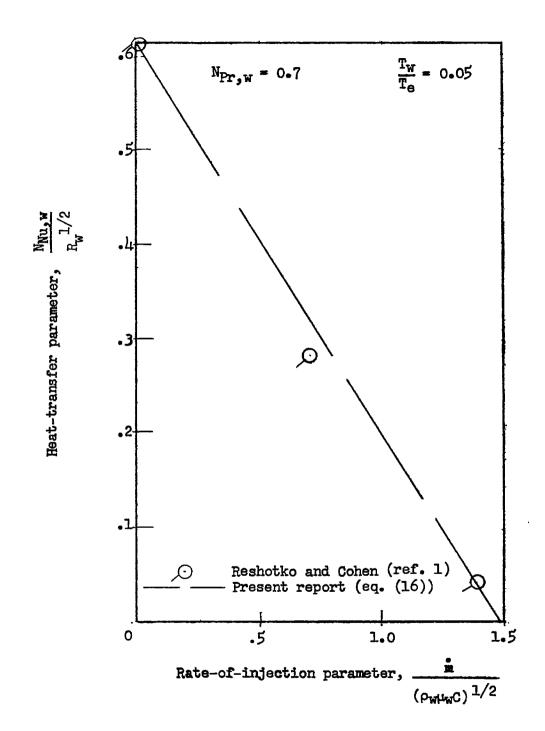
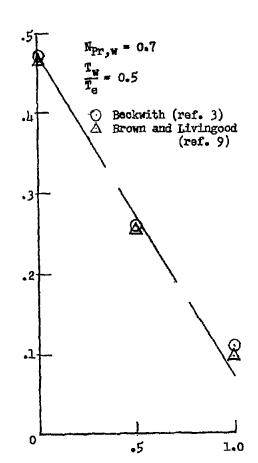Figure 3.- Effect of air-to-air injection on heat-transfer parameter.

(a) Axisymmetric case.

Figure 4.- Effect of air-to-air injection on heat-transfer parameter.

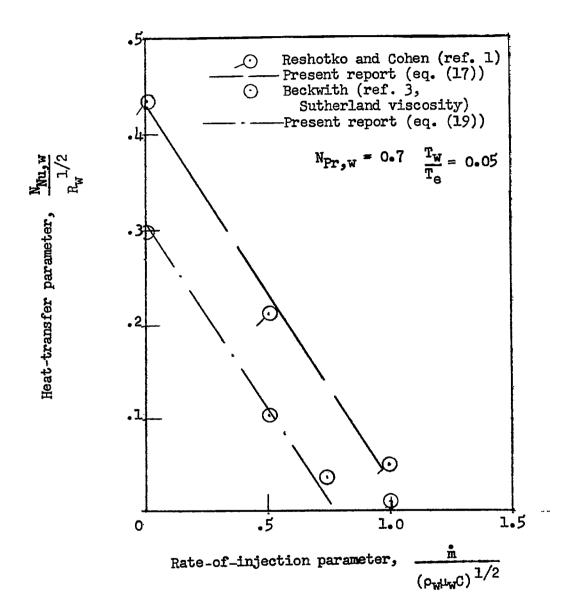(b) Two-dimensional case.

Figure 4.- Concluded.

Figure 5.- Effect of air-to-air injection on heat-transfer parameter
for two-dimensional case.

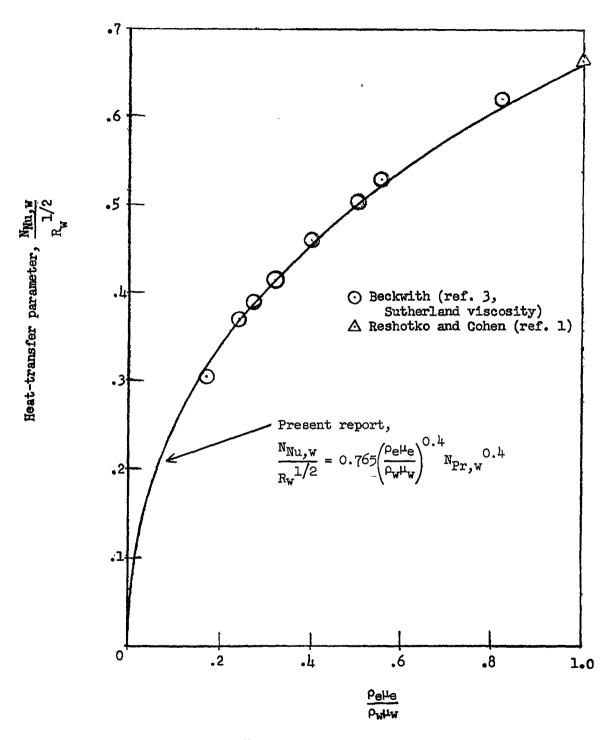Figure 6.- Variation of $\dfrac{N_{Nu,w}}{R_w^{1/2}}$ with $\dfrac{\rho_e\mu_e}{\rho_w\mu_w}$. Axisymmetric stagnation point; no injection.
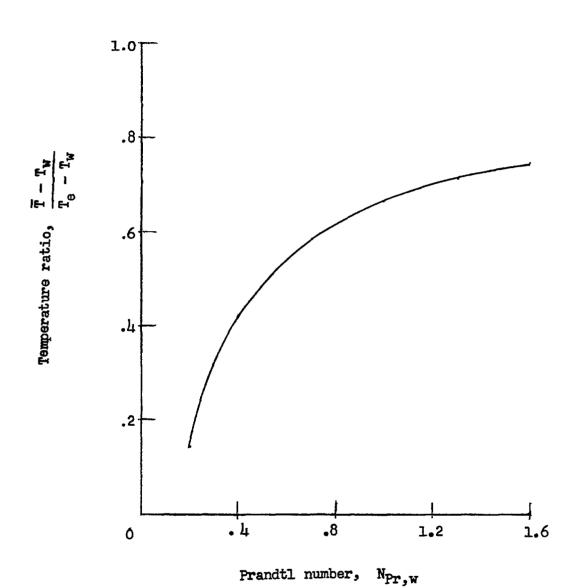
Figure 7.- Variation of $\dfrac{\overline{T} - T_w}{T_e - T_w}$ with $N_{Pr,w}$.
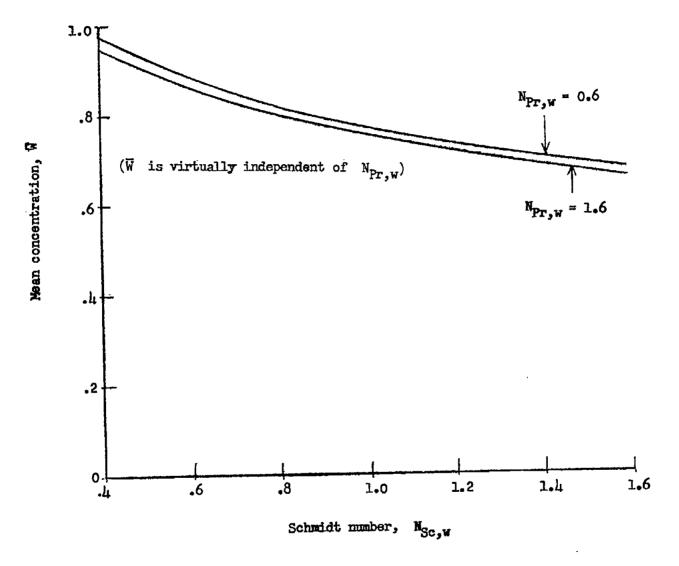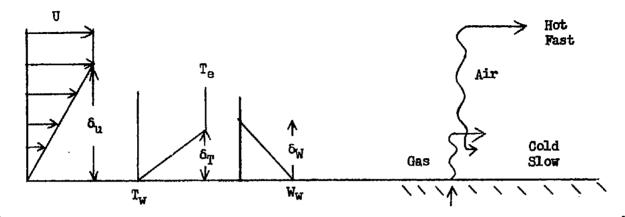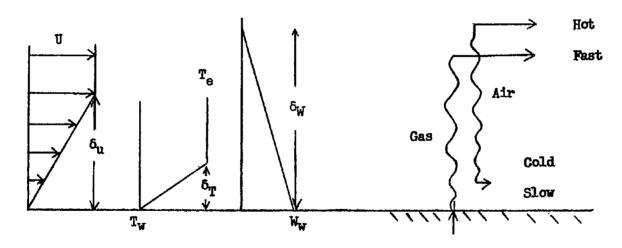
Figure 8.– Variation of $\overline{W}$ with $N_{Sc,w}$.

(a) $c_{p,2} > c_{p,1}$; maximum shielding when $N_{Pr,w}$ is large and $N_{Sc,w}$ is large.

(b) $c_{p,1} > c_{p,2}$; maximum shielding when $N_{Pr,w}$ is large and $N_{Sc,w}$ is small.

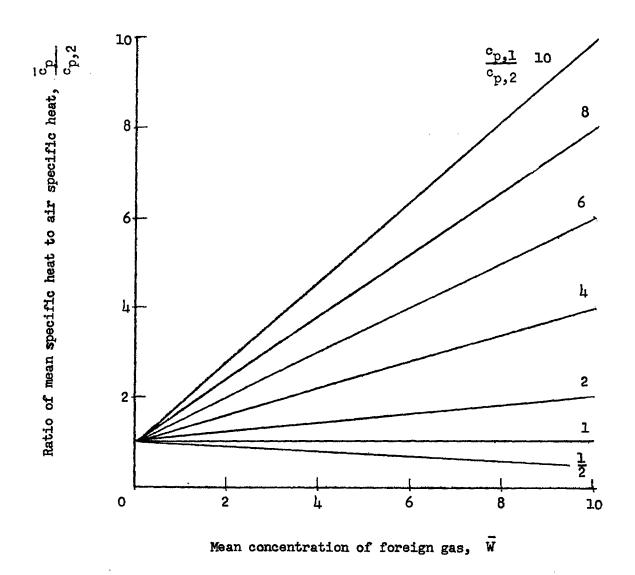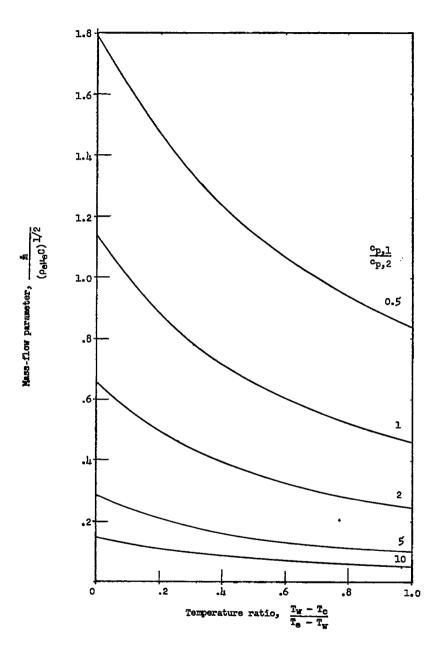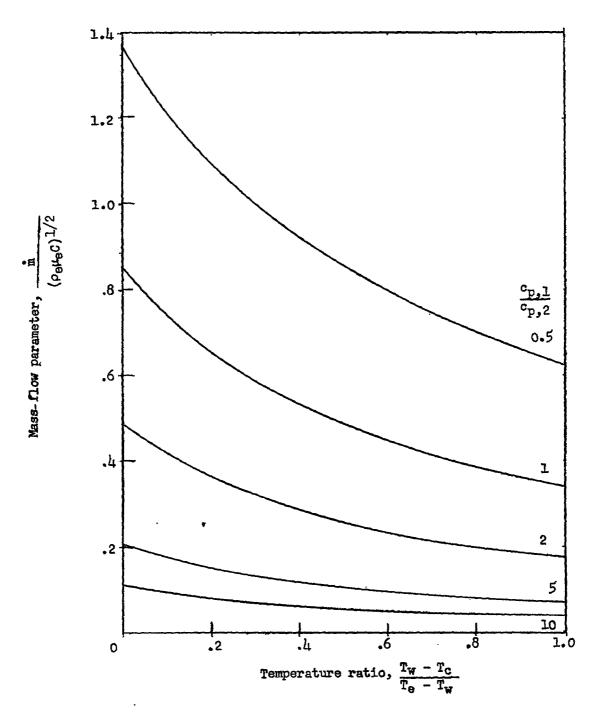Figure 9.— Qualitative description of the shielding mechanism.

Figure 10.- Variation of $\dfrac{\bar{c}_p}{c_{p,2}}$ with $\bar{W}$ for various values of $\dfrac{c_{p,1}}{c_{p,2}}$.

(a) Axisymmetric flow.

Figure 11.- Variation of mass-flow parameter with $\dfrac{T_w - T_c}{T_e - T_w}$ for various values of $\dfrac{c_{p,1}}{c_{p,2}}$.

(b) Two-dimensional flow.

Figure 11.- Concluded.